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On Solutions of $\Delta u = f(u)$

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Joseph B. Keller

Abstract

A simple upper bound is obtained for any solution of the nonlinear equation

$$\Delta u = f(u)$$

for a certain class of functions f . From this the nonexistence of entire solutions follows. The existence of solutions with infinite boundary values is also proved. Finally necessary and sufficient conditions on f for the existence of spherically symmetric entire solutions are obtained.

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1. Introduction

In this paper a simple upper bound is obtained for any solution, in any number of variables, of the nonlinear equation

$$(1) \quad \Delta u = f(u).$$

The bound depends upon the function $f(u)$, which must be positive and satisfy a certain growth condition, G. This result is stated in Theorem I. If $f(u)$ is negative then a lower bound is obtained instead. An interesting consequence of Theorem I is that for those $f(u)$ for which the bound applies, (1) has no entire solution, i.e., no solution regular in the full n -dimensional space. This conclusion, expressed in Theorem II, improves similar results of H. Wittich^[1], E. K. Haviland^[2] and W. Walter^[7] which apply to a more restricted class of functions $f(u)$ than those considered here. R. Osserman^[8] has recently obtained our result. Another consequence of Theorem I is the existence, in any bounded domain, of a solution of (1) which becomes infinite everywhere on the boundary of the domain, provided that $f(u)$ is an increasing function. This result is our Theorem III. It generalizes similar results of L. Bieberbach^[3] and H. Rademacher^[4], who considered the equation $\Delta u = e^u$ in two and three dimensions, respectively. Theorem IV states necessary and sufficient conditions on $f(u)$ for the existence of a spherically symmetric entire solution of (1).

The above results were obtained in 1954 in connection with the analysis of a physical problem^[5]. Some of this work was not relevant to the original problem, and is given here for the first time; some results were stated and used in [5] but the proofs were omitted.

It is a pleasure to thank Lipman Bers, Bernard Friedman and Louis Nirenberg for their advice and suggestions concerning this work.

2. Theorem I and its consequences

We begin by stating the growth condition G:

$f(u)$ satisfies condition G if it is a single-valued real continuous function defined for all real values of u and if there exists a positive non-decreasing continuous function $h'(u)$ such that $f(u) \geq h'(u)$ and

$$(2) \quad \int_0^{\infty} \left[\int_0^x h'(z) dz \right]^{-1/2} dx < \infty.$$

Now we may state Theorem I as follows:

Theorem I. Let $f(u)$ satisfy condition G and let u be a solution of (1) in a domain D of Euclidean n -space and continuous on its boundary S . Then there exists a decreasing function $g(R)$ determined by $h'(u)$ such that

$$(3) \quad u(P) \leq g[R(P)].$$

Here P denotes a point in D and $R(P)$ denotes its distance from S . The function $g(R)$ has the limits

$$(4) \quad g(R) \rightarrow \infty \quad \text{as} \quad R \rightarrow 0$$

$$(5) \quad g(R) \rightarrow -\infty \quad \text{as} \quad R \rightarrow \infty.$$

In the proof of Theorem I, given in Section 3, we will determine the function $g(R)$. However, we first show how several other theorems follow from Theorem I.

Theorem II. If $f(u)$ satisfies condition G, then (1) has no entire solution.

Proof. Suppose that there were such a solution. Then at any point P , $u(P) \leq g(R)$ for all R , since u is defined in a sphere of any radius R centered at P . But

then from (4) and (5), $u(P) \leq z$ for any number z . This is impossible, so there can be no such solution u , and the theorem is proved.

H. Wittich^[1] obtained this conclusion for two dimensions under the more restrictive condition that $f(u)$ be positive, convex (i.e., $f'' > 0$) and satisfy (2). E.K. Haviland^[2] extended the result to three dimensions with the additional requirement that $f(u) \geq c > 0$. His method can be applied in any number of dimensions. W. Walter^[7] eliminated the restriction $f(u) \geq c > 0$ and convexity in Haviland's result, but required f to be bounded below by a positive convex function which satisfied (2). R. Osserman^[8] proved the result under the same conditions we impose. R.M. Redheffer^[9] has announced (in an abstract) that in two dimensions the conclusion of Theorem II follows if f is merely positive, continuous and satisfies (2). This result is proved below (Theorem IV) for spherically symmetric solutions in any number of dimensions.

Another consequence of Theorem I, which we will now deduce, is

Theorem III. If $f(u)$ is nondecreasing and satisfies condition G, then in any bounded domain D there exists a solution of (1) which becomes infinite on S .

To prove this we note that for any constant α and any domain D there exists in D a solution u_α of (1) which is equal to α on S , provided that $f(u)$ is nondecreasing (Frank-von Mises^[6], p. 788). Furthermore, at each point of D , u_α increases with α ([6], p. 787). If $f(u)$ satisfies condition G then Theorem I holds, and at each point P in D all of the u_α are bounded above. Thus in every closed subdomain of D not containing points of S , as α becomes infinite the u_α converge uniformly to a limit u . This limit is also a solution of (1) ([6], p. 787). As P approaches S , $u(P)$ increases indefinitely, since on S , $u_\alpha = \alpha$ becomes infinite. Thus u is the desired solution and Theorem III is

proved.

In concluding this section we note that when $f(u)$ is negative, we may consider (1) as an equation for $v = -u$, and then $f(u)$ is replaced by $-f(-v)$, which is positive. Consequently the preceding theorems are valid for this case also, with the above modifications.

3. Proof of Theorem I.

Each point P of D is the center of a sphere of radius $R(P)$ which lies in D . Therefore it suffices to prove the theorem for D a sphere of radius R with P at its center. Thus suppose u is a solution of (1) in a sphere D of radius R , and suppose u is defined continuously on S . We define a function v in D and on S as the solution of the equations

$$(6) \quad \Delta v = h(v) \quad \text{in } D$$

$$(7) \quad v = \alpha \quad \text{on } S.$$

In (6), $h(v) = \theta h'(v)$ where $h'(v)$ is the function occurring in condition G and θ is a constant, $0 < \theta < 1$. Thus $f(u) > h(u)$. In (7) α is a constant satisfying

$$(8) \quad \alpha \geq u \quad \text{on } S.$$

The existence and uniqueness of a solution v of (6) and (7) is assured because h is a nondecreasing function ([6], pp. 787-8).

From (1) and (6) we have

$$(9) \quad \Delta(v-u) = [h(v) - h(u)] + [h(u) - f(u)].$$

Since $v-u \geq 0$ on S by (7), and since $v-u$ satisfies (9), we may conclude that $v-u$ is nowhere negative in D . For if $v-u$ were negative at any point in D , it

would have a negative minimum in D . At this minimum, from (9), we would have $\Delta(v-u) < 0$ since the first bracketed term on the right side has the same sign as $v-u$ or is zero and the second term is always negative. But this is impossible at a minimum. Therefore $v-u \geq 0$ in D or

$$(10) \quad v \geq u \quad \text{in } D.$$

We now define the function $g(R)$ by the equation

$$(11) \quad g(R) = \lim_{\alpha \rightarrow \infty} v(P).$$

Then, since v is an increasing function of α , we have, for every α , $v(P) \leq g(R)$. Combining this inequality with (10), we obtain

$$(12) \quad u(P) \leq g(R).$$

Equation (12) is the desired inequality (3) of Theorem I. It remains to be shown that $g(R)$ is finite, in order that (12) be non-trivial, and that $g(R)$ satisfies (4) and (5) and is a decreasing function of R .

To this end we must examine v , the solution of (6) and (7). First we see that v must be a function of r only, where r denotes distance from the center of the sphere. For if this were not the case, i.e., if v were not spherically symmetric, a different solution could be obtained by rotating v , and that would contradict the uniqueness of the solution. Therefore $v = v(r)$, and the equations satisfied by v become

$$(13) \quad v_{rr} + \frac{n-1}{r} v_r = h(v),$$

$$(14) \quad v_r(0) = 0,$$

$$(15) \quad v(R) = \alpha.$$

Equations (13) and (15) are just (6) and (7) written explicitly, and n denotes the dimension of the space. Equation (14) is a consequence of the regularity of v at $r = 0$.

Every real α uniquely determines $v(0)$ which is a monotonic increasing function of α . Therefore α is itself uniquely determined by $v(0)$. Thus we may replace (15) by

$$(16) \quad v(0) = v_0.$$

As v_0 increases, $\alpha = v(R)$ increases. We will show that $\alpha = v(R)$ is infinite for some value of v_0 . This value of v_0 is the $\lim_{\alpha \rightarrow \infty} v_0$, which was defined before as $g(R)$.

It is convenient to rewrite (13) in the form

$$(17) \quad (r^{n-1} v_r)_r = r^{n-1} h(v).$$

Integrating (17) from 0 to r yields

$$(18) \quad v_r(r) = r^{1-n} \int_0^r x^{n-1} h[v(x)] dx.$$

From (18) we see that $v_r \geq 0$. Therefore v is a nondecreasing function so we can obtain from (18)

$$(19) \quad v_r(r) \leq r^{1-n} h[v(r)] \int_0^r x^{n-1} dx = \frac{r}{n} h[v(r)].$$

Upon inserting (19) into (13) we get

$$(20) \quad v_{rr} \geq \frac{h[v]}{n}.$$

Since $v_r \geq 0$, (13) also yields $v_{rr} \leq h[v]$. Combining this with (20) leads to

$$(21) \quad h[v] \geq v_{rr} \geq \frac{h[v]}{n}.$$

We now multiply (21) by v_r and integrate from 0 to r to obtain

$$(22) \quad H(v, v_0) \geq v_r^2 \geq \frac{1}{n} H(v, v_0).$$

Here we have introduced $H(v, v_0)$ which is defined by

$$(23) \quad H(v, v_0) = 2 \int_{v_0}^v h(z) dz.$$

Next we take the square root of the reciprocal of each term in (21) and integrate again, obtaining

$$(24) \quad \int_{v_0}^v H^{-1/2}(z, v_0) dz \leq r \leq \sqrt{n} \int_{v_0}^v H^{-1/2}(z, v_0) dz.$$

By (2) the integral in (24) converges as v becomes infinite when $v_0 = 0$. But then the integral also converges for any value of v_0 . If we denote its limit by $A(v_0)$, (24) yields

$$(25) \quad A(v_0) \leq r \leq \sqrt{n} A(v_0),$$

where

$$(26) \quad A(v_0) = \int_{v_0}^{\infty} H^{-1/2}(z, v_0) dz.$$

From (25) we see that for each v_0 , v becomes infinite at a finite value of r in the range indicated in (25). Let us denote this value of r by $R(v_0)$.

The function $R(v_0)$ is continuous and non-increasing. If it were increasing then two solutions corresponding to different values of v_0 would have to be equal at some value of r . But this is impossible because a solution of (13) with a prescribed value on the surface of a sphere is unique.

Furthermore the integral $A(v_0)$ tends to $+\infty$ as v_0 tends to $-\infty$ and to zero as v_0 tends to $+\infty$. Therefore by (25) $R(v_0)$ behaves in the same way. We now define $g(R)$ as the 'inverse' of $R(v_0)$, i.e. $g(R) = \min \{ v_0 | R(v_0) = R \}$. This function is the desired $g(R)$ of Theorem I. It is decreasing and satisfies (4) and (5). This completes the proof of Theorem I.

4. Spherically symmetric solutions

In this section we will prove

Theorem IV. Let $f(u)$ be a real continuous function defined for all real u . Then (1) has an entire spherically symmetric solution if and only if one of the following conditions is satisfied:

- a) $f(u_1) = 0$
- b) $f(u) > 0$ and $\int_{u_0}^{\infty} F^{-1/2}(z) dz = \infty$
- c) $f(u) < 0$ and $\int_{-\infty}^{u_0} F^{-1/2}(z) dz = \infty$

In condition (a) u_1 is some constant, while in (b) and (c), $F(z)$ is defined by

$$(27) \quad F(z) = \int_{u_0}^z f(t) dt.$$

It is to be noted that the integrals in (b) and (c) diverge for all values of the constant u_0 in (27) if they diverge for any one value of it.

We will first show that if one of the above conditions is satisfied, then (1) has an entire solution. In case (a), the constant solution $u = u_1$ is such a solution. In cases (b) and (c) u must satisfy (13), with f in place of h , (14) and (16), all v 's being replaced by u 's in these equations. Let us first consider case (b). Then from (18), with f in place of h , it follows that $v_r \geq 0$. Now (13) yields

$$(28) \quad u_{rr} \leq f(u).$$

Upon multiplying (28) by u_r and integrating from $r = 0$ we obtain

$$(29) \quad u_r^2 \leq 2F(u)$$

From (29) we find

$$(30) \quad \frac{1}{\sqrt{2}} \int_{u_0}^u F^{-1/2}(z) dz \leq r.$$

Equation (30) and condition (b) show that u is finite if r is finite, while $u \rightarrow \infty$ as $r \rightarrow \infty$. Equation (29) shows that u_r is finite for r finite. Therefore the solution u is entire. Similar considerations apply in case (c), and thus we have shown that when any one of the conditions of the theorem is satisfied, (1) has an entire spherically symmetric solution.

Now we will prove the converse - that if none of the conditions is satisfied (1) has no entire spherically symmetric solution. If condition (a) is not satisfied, f is never zero, so it is either everywhere positive or everywhere negative. Let us first suppose that f is everywhere positive. Then if condition (b) is not satisfied we have

$$(31) \quad \int_{u_0}^{\infty} F^{-1/2}(z) dz \equiv B < \infty.$$

We must show that the solution of (13), (14) and (15) is not entire. By using (29) in (13) we obtain

$$(32) \quad u_{rr} \geq f(u) - \frac{(n-1)\sqrt{2}}{r} F^{1/2}(u).$$

If we multiply (32) by u_r and integrate from r_1 to r we obtain

$$(33) \quad u_r^2 \geq 2F(u) - 2F(u_1) - \frac{2(n-1)\sqrt{2}}{r_1} \int_{u_1}^u F^{1/2}(v) dv + u_r^2(r_1) \quad r \geq r_1.$$

Since $F(u)$ is an increasing function, we have

$$(34) \quad \int_{u_1}^u F^{1/2}(v) dv < F(u) \int_{u_1}^u \frac{dv}{F^{1/2}(v)} < F(u)B \quad u > u_1.$$

Using (34) in (33) yields

$$(35) \quad u_r^2 \geq 2F(u) \left[1 - \frac{(n-1)\sqrt{2} B}{r_1} \right] + u_r^2(r_1) - 2F(u_1). \quad r \geq r_1$$

For $r_1 = \varepsilon(n-1)\sqrt{2} B$, with $0 < \varepsilon < 1$ the coefficient of $F(u)$ in (35) is positive.

Then for r larger than some number $R > r_1$ the right side of (35) is positive, since $F(u)$ becomes infinite as u does (as we see from (31)) and u becomes infinite as r does we see by integrating (18). Thus we have from (35)

$$(36) \quad \int_{u(R)}^u \left\{ 2F(u)[1-\varepsilon] + u_r^2(r_1) - 2F(u_1) \right\}^{-1/2} du \geq r - R \quad r \geq R > r_1.$$

Since the integral on the left converges according to (31), u becomes infinite at a finite value of r . Thus there is no entire solution of (1) when $f > 0$ and the integral in (b) converges. Similar considerations apply for $f < 0$ and the integral in (c) convergent. Thus the proof is complete. This proof is similar, in part, to that of E. K. Haviland^[2].

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